Cybernetics and Systems Analysis, Vol. 44, No. 5, 2008

ON THE PROBLEM OF GUIDANCE OF AN AUTONOMOUS CONFLICT-CONTROLLED SYSTEM ONTO A CYLINDER SET[1]

Yu. V. Averboukh

UDC 517.977.8

The problem of guidance of a conflict-controlled system onto a cylinder set is considered. This problem is compared with the problem of guidance of a transformed system onto the base of the cylinder at the last moment. It is shown that the problems considered are equivalent. Moreover, the sequences constructed by the programmed iteration method coincide for both problems.

Keywords: *differential game, maximal stable bridge, "at a moment" problem, programmed iteration method.*

INTRODUCTION

This work is devoted to the investigation of the structure of the solution of a differential guidance game in the case when the goal set is of the form of a cylinder in the space of positions. Problems of this form are sometimes called "at a moment" guidance problems. The structure of a nonlinear guidance game problem is exhaustively characterized by the theorem on the alternative proved by N. N. Krasovskii and A. I. Subbotin [1, 2]; it asserts the existence of a saddle point in the class of corresponding positional strategies under conditions of information consistency. In the case when the condition of information consistency is not satisfied, a saddle point exists in the class of pairs "counterstrategy-strategy" [2] (the existence of a saddle point in pairs "strategy-counterstrategy" and "mixed strategy-mixed strategy" [2] is also established). The following form of an optimal strategy in the guidance problem [2] is well-known: a strategy (or a counterstrategy in the case when the condition of information consistency is not satisfied) is constructed by the method of extremal shift to some set; this set is a maximal u-stable bridge in the sense of N. N. Krasovskii. Thus, the solution of the guidance game problem is reduced to the problem of construction of the maximal u-stable bridge. If the problem is considered in the classes "mixed strategy-mixed strategy" and "strategy-counterstrategy," then the corresponding optimal control can be obtained by the method of extremal shift to the maximal $\widetilde{u}$-stable and u_*-stable bridge, respectively [2].

Along with guidance problems, game problems of minimization of a functional are considered in the theory of differential games. For these problems, based on the theorem on the alternative, N. N. Krasovskii and A. I. Subbotin established the existence of a function of price [2].

A concrete positional absorption set or a function of price can be constructed on the basis of the programmed iteration method proposed by A. G. Chentsov [3–6] (see also [7–9]). The following two important classes are often selected from the set of problems considered in the theory of differential games: "at the moment" problems and "at a moment" problems. A guidance game problem is called an "at the moment" problem if a system should approach a set in the phase space at the last moment and an "at a moment" problem if the system should approach a set in the phase space at any moment. In the latter case, the goal set of the game is of the form a cylinder in the space of positions. Similarly, a game problem of minimization

[1]This work is supported by the Russian Foundation for Fundamental Research, grants Nos. 06-01-00414 and 07-01-96088-p.

Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Science, Ekaterinburg, Russia, *ayv@imm.uran.ru*. Translated from Kibernetika i Sistemnyi Analiz, No. 5, pp. 117–126, September–October 2008. Original article submitted December 14, 2006.

1060-0396/08/4405-0729 ©2008 Springer Science+Business Media, Inc.

of a functional is called an "at the moment" problem if the value of the functional should be minimized at the last moment and an "at a moment" problem if the minimal value of a functional along a trajectory should be minimized. Game problems of minimization of a functional "at the moment" and "at a moment" can be investigated by the methods of the theories of generalized solutions of Hamilton–Jacobi equations and minimax inequalities, respectively, that are developed in [10, 11]. For the case that generalizes the case of games with simple motions, explicit expressions for the function of price for an "at the moment" problem and for an "at a moment" problem [10] are well-known. We note that an expression for the function of price in the case of a game with simple motions is first obtained in [12]. It is also well known that the solution of an "at a moment" problem for autonomous systems can be obtained from the solution of the Hamilton–Jacobi equation with the transformed Hamiltonian that is the minimum out of the Hamiltonian of the initial game and zero [13].

The solution of a lot of irregular linear "at the moment" problems is obtained owing to the programmed iteration method [3], but the solution of similar "at a moment" problems seems to be more complicated. A variant of representation of an "at a moment" problem in terms of an "at the moment" problem without any transformation of the system being controlled is presented in [3, p. 228–231].

In this work, problems are transformed using an original extension of systems being controlled. It is proved that, for the case of an autonomous conflict-controlled system, an "at a moment" guidance problem is equivalent to an "at the moment" guidance problem for the corresponding extended problem. The equivalence of the considered problems follows from the fact that the iterations constructed according to a variant of the programmed iteration method coincide for both problems. The considered method of transforming problems can be applied to game problems of minimization of trajectory functionals. In particular, the statements proved in this work imply that the "at a moment" problem for a conflict-controlled linear system with constant coefficients is equivalent to the "at the moment" problem for a bilinear conflict-controlled system.

FORMULATION OF THE MAIN RESULT

We consider approach-evasion problems with a set of M autonomous conflict-controlled systems of the form

$$\dot{x} = f(x,u,\upsilon), \quad u \in P, \quad \upsilon \in Q, \tag{1}$$

over a time interval $[0,\vartheta]$. It is assumed that the first player dealing with a control u strives to guide a system onto a set M, $M \subset [0,\vartheta]\times\mathbb{R}^n$, and the second player who deals with a control υ strives to prevent this.

It is assumed that f is continuous, locally Lipshitz with respect to the phase variable, and satisfies the condition of sublinear growth, $P \subset \mathbb{R}^p$, $Q \subset \mathbb{R}^q$, P and Q are compacts, and M is closed. We consider the obtained differential game in the class "counterstrategy-strategy." We consider that the first player (dealing with the control u) realizes control in the class of counterstrategies and that the second player (dealing with the control υ) realizes control in the class of pure positional strategies [2]. Following the definitions introduced in [2] for the general case, we will give strict definitions of a counterstrategy, a positional strategy, and a motion for the case of autonomous games.

By a counterstrategy of the first player we understand any function U: $[0,\vartheta]\times\mathbb{R}^n\times Q \to P$ measurable with respect to the third argument and by a positional strategy of the second player we understand any function V: $[0,\vartheta]\times\mathbb{R}^n \to Q$. The motion that is generated by a counterstrategy $U(t,x,\upsilon)$ over an interval $[t_*,t^*]$ and starts with a point x_*, following [2], we define as the limits of Euler's broken lines when the fineness of partitioning tends to zero. Euler's broken lines are constructed as follows. We assume that $\Delta = \{\tau_k\}_{k=0}^{r}$ is a partition of the interval $[t_*,t^*]$ and that $\upsilon(\cdot)$ is some control of the second player, and, on each interval $[\tau_{k-1},\tau_k)$ ($k=\overline{1,r}$), define Euler's broken line as the solution of the equation

$$x_k[t] = x_{k-1}[\tau_{k-1}] + \int_{\tau_{k-1}}^{t} f(x_k[\theta], U(\tau_{k-1}, x_{k-1}[\tau_{k-1}], \upsilon(\theta)), \upsilon(\theta))\, d\theta, \quad x_0[\tau_0] \triangleq x_*.$$

In this case, the second player forms the control according to the following rule: let $\Xi = \{\xi_j\}_{j=0}^{m}$ be some partition of the interval $[t_*,t^*]$, and let he choose his control unaltered over an interval $[\xi_{j-1},\xi_j)$, $j=\overline{1,m}$, and equal to $V(\xi_{j-1},x_{j-1})$, where x_{j-1} is the system position realized by the time ξ_{j-1}.

When the Isaacs condition (the condition of a saddle point in a small game)

$$\forall s,x \in \mathbb{R}^n \quad \min_{u\in P}\max_{v\in Q}\langle s, f(x,u,v)\rangle = \max_{v\in Q}\min_{u\in P}\langle s, f(x,u,v)\rangle$$

is true, it suffices to consider a differential game in the class of positional strategies [2], and then the strategies of the first player depend only on the position obtained. By the theorem on the alternative [2, Theorem 82.2], the solution of an approach-evasion problem is completely determined by the set of successful solvability (the maximal u-stable bridge) $\mathfrak{W}$. In this case, the optimal counterstrategy $U(t,x,v)$ is constructed according to the rule of extremal shift to $\mathfrak{W}$ [2].

We next consider the following two differential games over an interval $[0,\vartheta]$ in $\mathbb{R}^n$:

$$\dot{x} = f_1(x,u,v), \quad u \in P_1, \quad v \in Q_1;$$

$$\dot{x} = f_2(x,u,v), \quad u \in P_2, \quad v \in Q_2.$$

For each system, we consider the problem of its guidance onto sets M_1 and M_2, respectively (the sets M_i are closed). We consider that the problems of guidance of the ith system onto the set of M_i are equivalent (in the sense of N. N. Krasovskii and A. I. Subbotin) if the sets of successful solvability of both problems $\mathfrak{W}_i$ coincide.

The main result of this work is the assignment of equivalent differential "at the moment" games to some differential "at a moment" games. Let us consider the problem of guidance of system (1) onto a cylindrical set $[0,\vartheta]\times F$, where F is a closed subset of $\mathbb{R}^n$. As has been already noted, it is assumed that $P\subset\mathbb{R}^p$, $Q\subset\mathbb{R}^q$, and P and Q are compacts. This problem is an "at a moment" problem of guidance onto F. The mentioned conditions are imposed on the choice of the function $f(\cdot,\cdot,\cdot)$.

In order to construct an equivalent "at the moment" game, we extend the capabilities of the first player who forms the control u by introducing an additional control u_0 that assumes the value 0 or 1. Let us consider a conflict-controlled system

$$\dot{x} = u_0 f(x,u,v), \quad x \in \mathbb{R}^n, \quad u_0 \in \{0,1\}, \quad u \in P, \quad v \in Q, \tag{2}$$

over the interval $[0,\vartheta]$. In new system (2), the first player deals with the controls u_0 and u and the second player, as in system (1), deals with the control v. Similar methods of transformation of such a system were considered earlier in the class of program controls [13, 14]. We put the problem of guidance onto the set $\{\vartheta\}\times F$. This problem is an "at the moment" guidance problem. We prove that the problems being considered are equivalent. Moreover, we prove that the sequences constructed by one of programmed iteration methods (iterations for stability) coincide. Let us describe this variant of the programmed iteration method for the systems being considered [15].

In what follows, we will use the following denotations:

$$P^{(1)} \triangleq P, \quad P^{(2)} \triangleq \{0,\,1\}\times P, \quad M^{(1)} \triangleq [0,\,\vartheta]\times F, \quad M^{(2)} \triangleq \{\vartheta\}\times F;$$

for $x\in\mathbb{R}^n$, $v\in Q$, and $u\in P^{(1)}=P$, we put

$$f^{(1)}(x,u,v) \triangleq f(x,u,v);$$

for $x\in\mathbb{R}^n$, $v\in Q$, $u\in P^{(2)}$, $u=(u_0,u')$, $u_0\in\{0,\,1\}$, and $u'\in P$, we put

$$f^{(2)}(x,u,v) \triangleq u_0 f(x,u',\,v).$$

We denote by $\mathfrak{W}^{(1)}$ the set of successful solvability of the problem of guidance of system (1) onto $[0,\vartheta]\times F$ and by $\mathfrak{W}^{(2)}$ the set of successful solvability of the problem of guidance of system (2) onto $\{\vartheta\}\times F$.

For a fixed $v\in Q$, we consider the following controlled v-systems:

$$\dot{x} = f_v^{(i)}(x,u) \triangleq f^{(i)}(x,u,v), \quad x\in\mathbb{R}^n, \quad u\in P^{(i)},$$

over the time interval $[0,\vartheta]$ $(i=1,2)$. Following [3, 15], we introduce the set of all generalized program controls of the first player in the ith system over the time interval $[t_*,\vartheta]$, the set of all measures $[t_*,\vartheta]\times P^{(i)}$ matched with the

Lebesgue measure on $[t_*, \vartheta]$. We denote this set by $\mathcal{R}^{(i)}_{t_*}$. For each generalized program control $\mu^{(i)} \in \mathcal{R}^{(i)}_{t_*}$ and position (t_*, x_*), there is a unique solution of the equation

$$x(t) = x_* + \int\limits_{[t_*,\, t] \times P^{(i)}} f^{(i)}_{v}(x(\tau), u)\, \mu^{(i)}(d(\tau, u)).$$

We denote this solution by $\varphi^{(i)}(\,\cdot\,, t_*, x_*, \mu^{(i)}, v)$.

We introduce a variant of the programmed iteration method [3, p. 233], [15]. We assume that $E \subset [0, \vartheta] \times \mathbb{R}^n$ and that E is closed and put

$$A^{(i)}_{v}(E) \triangleq \{(t_*, x_*) \in E \colon \exists \mu^{(i)} \in \mathcal{R}^{(i)}_{t_*} \colon\ \exists \tau \in [t_*, \vartheta] \colon \varphi^{(i)}(\tau, t_*, x_*, \mu^{(i)}, v) \in M^{(i)}[\tau]\ \&$$

$$\forall t \in [t_*, \tau]\, \varphi^{(i)}(t, t_*, x_*, \mu^{(i)}, v) \in E[t]\}.$$

Here, $E[t]$ denotes a section of the set E, $E[t] \triangleq \{x \in \mathbb{R}^n \colon (t, x) \in E\}$. We note that $A^{(i)}_{v}(E)$ is also closed.

We put

$$A^{(i)}(E) \triangleq \bigcap_{v \in Q} A^{(i)}_{v}(E).$$

The operator $A^{(i)}$ is called the program absorption operator [3]. We construct the following sequences of sets:

$$W^{(i)}_0 \triangleq [0, \vartheta] \times \mathbb{R}^n, \quad W^{(i)}_k = A^{(i)}(W^{(i)}_{k-1}) \quad \forall k \in \mathbb{N}.$$

As is well known [15], we have

$$\mathfrak{W}^{(i)} = \bigcap_{k \in \mathbb{N}} W^{(i)}. \tag{3}$$

THEOREM. The following statements are true:

(1) $W^{(1)}_k = W^{(2)}_k$ for all $k \in \mathbb{N}$;

(2) the problem of guidance of system (1) onto the cylindrical set $[0, \vartheta] \times F$ is equivalent to the problem of guidance of system (2) onto the set $\{\vartheta\} \times F$;

(3) if system (1) satisfies the Isaacs condition, then system (2) also satisfies this condition.

Thus, even an "at a moment" linear game is equivalent to an "at the moment" bilinear game.

The concept of equivalence is naturally extended to games in which players strive to minimize-maximize a payoff functional. In this case, we call two games equivalent if their functions of price coincide.

COROLLARY. Let the dynamics of a system be described by Eq. (1), and let the first (second) player strive to minimize (maximize) the payoff functional of the form $\min\limits_{t \in [0, \vartheta]} \omega(x(t))$ (it is assumed that the function ω is continuous and satisfies the condition of sublinear growth). Then this differential game is equivalent to a game in which the dynamics of the system is described by Eq. (2) with the payoff functional $\omega(x(\vartheta))$.

SOME PROPERTIES OF PROGRAM ABSORPTION OPERATORS

We call a set E nonincreasing with respect to sections if $E[t^*] \subset E[t_*]$ for all t^*, $t_* \in [0, \vartheta]$ such that we have $t^* \geq t_*$.

LEMMA 1. If E is a closed set that does not increase with respect to sections and is such that we have $[0, \vartheta] \times F \subset E$, then $A^{(2)}(E)$ is also a set that does not increase with respect to sections and is such that we have $[0, \vartheta] \times F \subset E$.

Proof. Let $(t^*, x_*) \in A^{(2)}(E)$. We prove that $(t_*, x_*) \in A^{(2)}(E)$ for $t_* \leq t^*$. We fix $v \in Q$. There exists $\mu \in \mathcal{R}^{(2)}_{t^*}$ such that we have $\varphi^{(2)}(\vartheta, t^*, x_*, \mu, v) \in F$ and, for all $t \in [t^*, \vartheta]$, we have $\varphi^{(2)}(t, t^*, x_*, \mu, v) \in E[t]$.

We define a generalized control σ that is used in the second system and corresponds to the usual control $u_0 = 0$. Let $\hat{u} = (0, u_1, \dots u_p)$, where $(u_1, \dots u_p)$ is an arbitrary element P. For each measurable set $R \subset [0, \vartheta] \times P$, we put

$$\sigma(R) = \lambda\{t: (t, \hat{u}) \in R\}, \tag{4}$$

where λ is the Lebesgue measure on $[0, \vartheta]$. We define a measure $\hat{\mu}$ on $[t_*, \vartheta] \times P^{(2)}$ according to the following rule: with each measurable $R \subset [t_*, \vartheta] \times P^{(2)}$ we associate

$$\hat{\mu}(R) \triangleq \sigma(R \cap ([t_*, t^*) \times P^{(2)})) + \mu(R \cap ([t^*, \vartheta] \times P^{(2)})).$$

It may be noted that we have $\varphi^{(2)}(t, t_*, x_*, \hat{\mu}, \upsilon) = x_*$ when $t \in [t_*, t^*]$ and $\varphi^{(2)}(t, t_*, x_*, \hat{\mu}, \upsilon) = \varphi^{(2)}(t, t^*, x_*, \mu, \upsilon)$ when $t \in [t^*, \vartheta]$. Hence, we obtain

$$\varphi^{(2)}(\vartheta, t_*, x_*, \hat{\mu}, \upsilon) = \varphi^{(2)}(\vartheta, t^*, x_*, \mu, \upsilon) \in F,$$

$$\varphi^{(2)}(t, t_*, x_*, \hat{\mu}, \upsilon) = \varphi^{(2)}(t, t^*, x_*, \mu, \upsilon) \in E[t] \text{ for } t \in [t^*, \vartheta],$$

$$\varphi^{(2)}(t, t_*, x_*, \hat{\mu}, \upsilon) = x_* \in E[t^*] \subset E[t] \text{ for } t \in [t_*, t^*],$$

since E does not increase with respect to sections. Thus, we have $(t_*, x_*) \in A_\upsilon^{(2)}(E)$ for each $\upsilon \in Q$. From this we have $(t_*, x_*) \in A^{(2)}(E)$.

At the same time, if $[0, \vartheta] \times F \subset E$, then we also have $[0, \vartheta] \times F \subset E^{(2)}$. In fact, if we have $(t_*, x_*) \in [0, \vartheta] \times F$, then, for any $\upsilon \in Q$, we obtain $\varphi^{(2)}(t, t_*, x_*, \sigma, \upsilon) = x_* \in F \subset E[t]$. □

In addition to generalized controls, we also introduce program controls in the ith υ-system over the interval $[t_*, \vartheta]$, namely, measurable functions

$$\mathbf{u}^{(i)}: [t_*, \vartheta] \to P^{(i)}.$$

For each program control $\mathbf{u}^{(i)}(\cdot)$ and a position (t_*, x_*), there is a unique solution of each of equations $\dot{x} = f_\upsilon^{(i)}(x, \mathbf{u}^{(i)}(\cdot))$ with the initial data $x(t_*) = x_*$. We denote this solution by $x^{(i)}(\cdot, t_*, x_*, \mathbf{u}^{(i)}(\cdot), \upsilon)$. Note that the set of usual program controls is embedded into the set of generalized program controls with the help of an operator that associates with $\mathbf{u}^{(i)}(\cdot)$ a generalized control $\mu^{(i)}_{\mathbf{u}^{(i)}(\cdot)}$ according to the rule

$$\mu^{(i)}_{\mathbf{u}^{(i)}(\cdot)}(R) \triangleq \lambda\{t: (t, \mathbf{u}^{(i)}(t)) \in R\},$$

where λ is the Lebesgue measure on $[0, \vartheta]$.

The rule of extremal shift implies that, for any generalized control $\mu^{(i)} \in \mathcal{R}^{(i)}_{t_*}$ and any position (t_*, x_*), there is a sequence of piecewise-constant right-continuous program controls $\mathbf{u}_k^{(i)}(\cdot)$ such that $x^{(i)}(\cdot, t_*, x_*, \mathbf{u}_k^{(i)}(\cdot), \upsilon)$ converges to $\varphi^{(i)}(\cdot, t_*, x_*, \mu^{(i)}, \upsilon)$ uniformly on $[t_*, \vartheta]$ [16].

We assume that $u \in P^{(i)}$ $(i = 1, 2)$, $\upsilon \in Q$, $\tau \geq 0$, and denote by $S_{u,\upsilon}^{(i),\tau}(x)$ the mapping of the shift in time τ along the trajectory outgoing from the position $(0, x)$ generated by the constant controls u and υ in the ith system,

$$S_{u,\upsilon}^{(i),\tau}(x) \triangleq x^{(i)}(\tau, 0, x, u, \upsilon).$$

Since the systems being considered are autonomous, we have

$$S_{u,\upsilon}^{(i),\tau}(x) = x^{(i)}(\tau + t, t, x, u, \upsilon) \quad \forall t \in [0, +\infty).$$

Note that, for all $\upsilon \in Q$ and $\tau > 0$, the following properties take place:

(1) if we have $u \in P^{(2)}$ and $u = (0, u_1, \ldots, u_p)$, then we obtain

$$S_{u,v}^{(2),\tau} = I,$$

where I is the identity mapping onto $\mathbb{R}^n$;

(2) if $u \in P^{(2)}$ and $u = (1, u_1, \ldots, u_p)$, then we have

$$S_{u,v}^{(2),\tau} = S_{u',v}^{(1),\tau},$$

where $u' \triangleq (u_1, \ldots, u_p) \in P^{(1)}$.

We fix $v \in Q$. Let $\mathbf{u}^{(i)}(\cdot)$ be some piecewise-constant right-continuous function defined on $[t_*, t^*]$ with values in $P^{(i)}$. We denote by Δ a collection of positive numbers $\{\delta^k\}_{k=1}^m$ such that if we have $\zeta^0 = t_*$, $\zeta^k = \zeta^{k-1} + \delta^k$, $k = \overline{1,m}$, $\zeta^m = t^*$, then the value of $\mathbf{u}^{(i)}(\cdot)$ on $[\zeta^{k-1}, \zeta^k)$ is constant and equals u^k. In this case, for any $x_0 \in \mathbb{R}^n$ the following equality is true:

$$x^{(i)}(t^*, t_*, x_0, \mathbf{u}^{(i)}(\cdot), v) = S_{u^m, v}^{(i),\delta^m} \circ \ldots \circ S_{u^1, v}^{(i),\delta^1}(x_0). \tag{5}$$

Let us consider some piecewise-constant right-continuous control in the second system $\mathbf{u}^{(2)}(\cdot)$. As in the general case, we denote by $\Delta = \{\delta^k\}_{k=1}^m$ a collection of positive numbers such that if we have $\zeta^0 = t_*$, $\zeta^k = \zeta^{k-1} + \delta^k$, $k = \overline{1,m}$, and $\zeta^m = \vartheta$, then we obtain $\mathbf{u}^{(2)}(t) = u^k = (u_0^k, u_1^k, \ldots, u_p^k)$, $t \in [\zeta^{k-1}, \zeta^k)$, $k = \overline{1,m}$.

We construct the control based on the given function $\mathbf{u}^{(2)}$ in the first system as follows. We choose the numbers k_j for which $u_0^{k_j} = 1$ and, as a result, obtain a collection of numbers

$$J \triangleq \{k \in \Delta\colon u_0^k = 1\} = \{k_1, \ldots, k_r\}.$$

Assume that $\gamma^0 = t_*$ and $\gamma^j = \gamma^{j-1} + \delta^{k_j}$, $j = \overline{1,r}$. Let us denote

$$\tau \triangleq \gamma^{j_r}. \tag{6}$$

We define the function

$$\mathbf{u}^{(1)}(t) \triangleq \begin{cases} (u_1^{k_j}, \ldots, u_p^{k_j}), & t \in [\gamma^{j-1}, \gamma^j), \\ \widetilde{u}, & t \in [\tau, \vartheta], \end{cases} \tag{7}$$

where $\widetilde{u}$ is an arbitrary element P. Denote $\kappa^j \triangleq (u_1^{k_j}, \ldots, u_p^{k_j})$.

Let us define a mapping $\theta(\cdot)\colon [t_*, \tau] \to [t_*, \vartheta]$ according to the following rule:

$$\theta(t) \triangleq \zeta^{k_j - 1} + (t - \gamma^{j-1}) \text{ for } t \in [\gamma^{j-1}, \gamma^j), \quad \theta(\tau) \triangleq \vartheta. \tag{8}$$

The mapping θ "adds" the time intervals that have been cut out in the definition of $\mathbf{u}^{(1)}$.

The lemma presented below is true.

LEMMA 2. For all $v \in Q$, $x_* \in \mathbb{R}^n$, $t_* \in [0, \vartheta]$, and piecewise-constant right-continuous functions $\mathbf{u}^{(2)}(\cdot)\colon [t_*, \vartheta] \to P^{(2)}$, the following equality is true:

$$x^{(1)}(t, t_*, x_*, \mathbf{u}^{(1)}(\cdot), v) = x^{(2)}(\theta(t), t_*, x_*, \mathbf{u}^{(2)}(\cdot), v) \quad \forall t \in [t_*, \tau].$$

The function $\mathbf{u}^{(1)}(\cdot)$ is defined by expression (7) and the function $\theta(\cdot)$ is defined by the rule (8).

Proof. We fix $t \in [t_*, \tau]$. There is j such that we have $t \in [\gamma^{j-1}, \gamma^j)$. Hence (see (8)), we have $\theta(t) \in [\zeta^{k_j - 1}, \zeta^{k_j})$. The control $\mathbf{u}^{(2)}(\cdot)$ is constant on half-intervals $[\zeta^{l-1}, \zeta^l)$, $l = \overline{1, k_j - 1}$, and the half-interval $[\zeta^{k_j - 1}, \theta(t))$. Then we have (see (5) for $i = 2$)

$$x^{(2)}(\theta(t), t_*, x_*, \mathbf{u}^{(2)}(\cdot), v) = S^{(2), \theta(t) - \zeta^{k_j - 1}}_{u^{k_j}, v} \circ S^{(2), \delta^{k_j - 1}}_{u^{k_j - 1}, v} \circ \ldots \circ S^{(2), \delta^1}_{u^1, v}(x_*).$$

We obtain (see Property 1) $S^{(2), \delta^l}_{u^l, v} = I$ for all $l \notin J$. From this we have

$$x^{(2)}(\theta(t), t_*, x_*, \mathbf{u}^{(2)}(\cdot), v) = S^{(2), \theta(t) - \zeta^{k_j - 1}}_{u^{k_j}, v} \circ S^{(2), \delta^{k_{j-1}}}_{u^{k_{j-1}}, v} \circ \ldots \circ S^{(2), \delta^{k_1}}_{u^{k_1}, v}(x_*).$$

Since we have $u^{k_j} = (1, u_1^{k_j}, \ldots, u_p^{k_j})$, Property 2 implies that

$$x^{(2)}(\theta(t), t_*, x_*, \mathbf{u}^{(2)}(\cdot), v) = S^{(1), t - \gamma^{j-1}}_{\kappa^j, v} \circ S^{(1), \delta^{k_{j-1}}}_{\kappa^{j-1}, v} \circ \ldots \circ S^{(1), \delta^{k_1}}_{\kappa^1, v}(x_*).$$

Here, we have used the equality $\theta(t) - \zeta^{k_j - 1} = t - \gamma^{j-1}$ (see (8)). For $i = 1$, we obtain the following equality from equality (5):

$$x^{(1)}(t, t_*, x_*, \mathbf{u}^{(1)}(\cdot), v) = S^{(1), t - \gamma^{j-1}}_{\kappa^j, v} \circ S^{(1), \delta^{k_{j-1}}}_{\kappa^{j-1}, v} \circ \ldots \circ S^{(1), \delta^{k_1}}_{\kappa^1, v}(x_*).$$

This implies the statement of the lemma. $\square$

Let us compare images of the two program absorption operators being considered.

LEMMA 3. Let E be a closed set nonincreasing with respect to sections, $[0, \vartheta] \times F \subset E$. Then we have $A^{(1)}(E) = A^{(2)}(E)$.

Proof. We first prove the embedding $A^{(1)}(E) \subset A^{(2)}(E)$. Let $(t_*, x_*) \in A^{(1)}(E)$. Then, for each $v \in Q$, there is a generalized control $\mu^{(1)} \in \mathcal{R}^{(1)}_{t_*}$ possessing the following property: there is $\tau \in [t_*, \vartheta]$ such that we have $\varphi^{(1)}(\tau, t_*, x_*, \mu^{(1)}, v) \in F$ and $\varphi^{(1)}(t, t_*, x_*, \mu^{(1)}, v) \in E[t]$, $t \in [t_*, \tau]$. We define $\mu^{(2)} \in \Pi^{(2)}$ as follows: if $R \subset [t_*, \vartheta] \times P^{(2)} = [t_*, \vartheta] \times \{0, 1\} \times P$, then we have

$$\mu^{(2)}(R) \triangleq \mu^{(1)}(\{(t, u') \colon t \in [t_*, \tau], \ u' \in P, \ (t, 1, u') \in R\}) + \sigma(R \cap ((\tau, \vartheta] \times \{0, 1\} \times P)).$$

The measure σ is defined in the proof of Lemma 1 (see (4)).

It may be noted that we have

$$\varphi^{(2)}(t, t_*, x_*, \mu^{(2)}, v) = \varphi^{(1)}(t, t_*, x_*, \mu^{(1)}, v) \text{ for } t \in [t_*, \tau],$$

$$\varphi^{(2)}(t, t_*, x_*, \mu^{(2)}, v) = \varphi^{(2)}(\tau, t_*, x_*, \mu^{(2)}, v) \in F \text{ for } t \in [\tau, \vartheta].$$

Hence, since we have $F \subset E[t]$, $t \in [0, \vartheta]$, we obtain $(t_*, x_*) \in A^{(2)}_v(E)$. By virtue of arbitrariness of v, we have $(t_*, x_*) \in A^{(2)}(E)$ for all $(t_*, x_*) \in A^{(1)}(E)$.

We now prove the inverse embedding. Let $(t_*, x_*) \in A^{(2)}(E)$. Hence, for each $v \in Q$, there is $\mu^{(2)} \in \mathcal{R}^{(2)}_{t_*}$ possessing the following property $\varphi^{(2)}(\vartheta, t_*, x_*, \mu^{(2)}, v) \in F$ and $\varphi^{(2)}(t, t_*, x_*, \mu^{(2)}, v) \in E[t]$ for all $t \in [t_*, \vartheta]$. Let $\{\mathbf{u}_k^{(2)}(\cdot)\}_{k=1}^{\infty}$ be a sequence of piecewise-constant program controls that is such that the sequence of trajectories $x^{(2)}(\cdot, t_*, x_*, \mathbf{u}^{(2)}(\cdot), v)$ converges to $\varphi^{(2)}(\cdot, t_*, x_*, \mu^{(2)}, v)$. For each k, moments τ_k and functions $\mathbf{u}_k^{(1)}(\cdot)$ and $\theta_k(\cdot)$ are specified (see (6)–(8)). By the definition of functions $\theta_k(\cdot)$ and Lemma 2, we obtain

$$x^{(1)}(t, t_*, x_*, \mathbf{u}_k^{(1)}(\cdot), v) = x^{(2)}(\theta_k(t), t_*, x_*, \mathbf{u}_k^{(2)}(\cdot), v), \quad t \in [t_*, \tau_k], \tag{9}$$

and, at the same time, we have

$$\theta_k(\tau_k) = \vartheta. \tag{10}$$

Let

$$\varepsilon_k \triangleq \max_{t \in [t_*, \vartheta]} ||x^{(2)}(t, t_*, x_*, \mathbf{u}_k^{(2)}(\cdot), v) - \varphi^{(2)}(t, t_*, x_*, \mu^{(2)}, v)||. \tag{11}$$

It follows from equalities (9) and (10) that we have

$$\varepsilon_k \to 0, \quad k \to \infty. \tag{12}$$

Since

$$\varphi^{(2)}(\vartheta, t_*, x_*, \mu^{(2)}, v) \in F, \qquad \varphi^{(2)}(t, t_*, x_*, \mu^{(2)}, v) \in E[t] \quad \forall t \in [t_*, \vartheta],$$

and E is a set nonincreasing with respect to sections and such that we have $F \subset E[t]$ for all $t \in [0, \vartheta]$, we obtain

$$d(x^{(1)}(\tau_k, t_*, x_*, \mathbf{u}_k^{(1)}(\cdot), v), F) \le \varepsilon_k, \tag{13}$$

$$d(x^{(1)}(t, t_*, x_*, \mathbf{u}_k^{(1)}(\cdot), v), E[t]) \le \varepsilon_k, \quad t \in [t_*, \tau_k]. \tag{14}$$

Here, $d(x, A)$ is the distance from a point $x \in \mathbb{R}^n$ to a set $A \subset \mathbb{R}^n$.

There are a sequence $\{k_l\}_{l=1}^{\infty}$ and some $\hat{\tau} \in [t_*, \vartheta]$ such that $\{\tau_{k_l}\}$ converges to $\hat{\tau}$. Without any loss of generality, we can consider that $\{\tau_k\}$ itself converges to $\hat{\tau}$. We denote by $\mu^{(1)}$ the weak limit of some subsequence of a sequence $\mu^{(1)}_{\mathbf{u}_k^{(1)}}$. From formulas (9), (11 (12), (13), and (14) we obtain the following embeddings:

$$\varphi^{(1)}(\hat{\tau}, t_*, x_*, \mu^{(1)}, v) \in F, \quad \varphi^{(1)}(\hat{\tau}, t_*, x_*, \mu^{(1)}, v) \in E[t], \quad t \in [t_*, \hat{\tau}].$$

Hence, we have $(t_*, x_*) \in A_v^{(1)}(E)$. Since the choice of $v \in Q$ has been arbitrary, we have $(t_*, x_*) \in A^{(1)}(E)$ for all $(t_*, x_*) \in A^{(2)}(E)$. ◻

PROOF OF THE MAIN STATEMENT

Statement 1 of the theorem follows from Lemmas 1 and 3 since $[0, \vartheta] \times \mathbb{R}^n$ is a set nonincreasing with respect to sections. Statement 2 of the theorem follows from the previous one by virtue of formula (3).

Proof of Statement 3 of the theorem. Under the condition, for all $s, x \in \mathbb{R}^n$, we have

$$\max_{v \in Q} \min_{u \in P} \langle s, f(x, u, v) \rangle = \min_{u \in P} \max_{v \in Q} \langle s, f(x, u, v) \rangle.$$

Let us prove the equality

$$\max_{v \in Q} \min_{u_0 \in \{0,1\}, u \in P} \langle s, u_0 f(x, u, v) \rangle = \min_{u_0 \in \{0,1\}, u \in P} \max_{v \in Q} \langle s, u_0 f(x, u, v) \rangle. \tag{15}$$

We first consider the case when $\max_{v \in Q} \min_{u \in P} \langle s, f(x, u, v) \rangle \le 0$, and then, for all $v \in Q$, we have $\min_{u \in P} \langle s, f(x, u, v) \rangle \le 0$, whence, for all $v \in Q$, we obtain

$$\min_{u_0 \in \{0,1\},\, u \in P} \langle s, u_0 f(x, u, v) \rangle = \min_{u \in P} \langle s, f(x, u, v) \rangle.$$

Hence, we have

$$\max_{v \in Q} \min_{u_0 \in \{0,1\},\, u \in P} \langle s, u_0 f(x, u, v) \rangle = \max_{v \in Q} \min_{u \in P} \langle s, f(x, u, v) \rangle,$$

$$\max_{v \in Q} \langle s, u_0 f(x, u, v) \rangle = \begin{cases} 0, & u_0 = 0; \\ \max_{v \in Q} \langle s, f(x, u, v) \rangle, & u_0 = 1. \end{cases} \tag{16}$$

Taking into account the condition of negativity of $\min_{u \in P} \max_{v \in Q} \langle s, f(x,u,v) \rangle$, we obtain

$$\min_{u_0 \in \{0,1\},\, u \in P} \max_{v \in Q} \langle s, u_0 f(x,u,v) \rangle = \min_{u \in P} \max_{v \in Q} \langle s, f(x,u,v) \rangle.$$

Hence, for the case when $\max_{v \in Q} \min_{u \in P} \langle s, f(x,u,v) \rangle \le 0$, equality (15) is proved.

Now let we have

$$\max_{v \in Q} \min_{u \in P} \langle s, f(x,u,v) \rangle \ge 0.$$

For v for which the inequality $\min_{u \in P} \langle s, f(x,u,v) \rangle \ge 0$ is fulfilled, we have

$$\min_{u_0 \in \{0,1\}, u \in P} \langle s, u_0 f(x,u,v) \rangle = 0.$$

From this we obtain $\max_{v \in Q} \min_{u_0 \in \{0,1\},\, u \in P} \langle s, u_0 f(x,u,v) \rangle = 0.$

We also have $\max_{v \in Q} \langle s, f(x,u,v) \rangle \ge 0$ for all $u \in P$. From representation (16), we obtain

$$\min_{u_0 \in \{0,1\}, u \in P} \max_{v \in Q} \langle s, u_0 f(x,u,v) \rangle = 0 = \max_{v \in Q} \min_{u_0 \in \{0,1\}, u \in P} \langle s, u_0 f(x,u,v) \rangle. \qquad \square$$

Proof of the corollary. For each $c > 0$, we consider the set $F_c \triangleq \{x \in \mathbb{R}^n : \omega(x) \le c\}$. Then, for each $c > 0$, the problem of guidance of system (1) onto the set $[0, \vartheta] \times F_c$ is equivalent to the problem of guidance of system (2) onto the set $\{\vartheta\} \times F_c$.

Let us fix a position (t_0, x_0), then there is $\gamma = \gamma(t_0, x_0)$, i.e., the lower bound of c for which the problem of guidance of system (1) is solvable over $[0, \vartheta] \times F_c$. The equivalence of the problem of guidance of system (1) onto the set $[0, \vartheta] \times F_c$ and the problem of guidance of system (2) onto the set $\{\vartheta\} \times F_c$ implies that γ is the lower bound of c for which the problem of guidance of system (2) onto $\{\vartheta\} \times F_c$ is solvable. Thus, the functions of price coincide in both problems. $\square$

REFERENCES

1. N. N. Krasovskii and A. I. Subbotin, "An alternative for the game problem of motion," Prikl. Mat. Mekh., **34**, No. 6, 1005–1022 (1970).
2. N. N. Krasovskii and A. I. Subbotin, Positional Differential Games [in Russian], Nauka, Moscow (1974).
3. A. I. Subbotin and A. G. Chentsov, Guarantee Optimization in Control Problems [in Russian], Nauka, Moscow (1981).
4. A. G. Chentsov, "The structure of an approach game problem," Dokl. Akad. Nauk SSSR, **224**, No. 6, 1272–1275 (1975).
5. A. G. Chentsov, "A game problem on the approach at a given instant," Mat. Sb., **99**, No. 3, 394–420 (1976).
6. A. G. Chentsov, "On a guidance game problem," Dokl. Akad. Nauk SSSR, **226**, No. 1, 73–76 (1976).
7. S. V. Chistyakov, "On solving pursuit game problems," Prikl. Mat. Mekh., **41**, No. 5, 825–832 (1977).
8. A. A. Melikyan, "Game cost in a linear approach differential game," Dokl. Akad. Nauk SSSR, **237**, No. 3, 521–524 (1977).
9. V. I. Ukhobotov, "Construction of a stable bridge for a class of linear games," Prikl. Mat. Mekh., **41**, No. 2, 358–364 (1977).
10. A. I. Subbotin, Generalized Solutions of First-Order Differential Equations [in Russian], RKhD, Izhevsk (2003).
11. A. I. Subbotin, Minimax Inequalities and Hamilton–Jacobi Equations [in Russian], Nauka, Moscow (1991).
12. B. N. Pshenichnyi and M. I. Sagaidak, "Differential games with fixed time," Cybernetics, No. 2, 54–63 (1970).
13. I. M. Mitchel, A. M. Bayen, and C. J. Tomlin, "A time-depend Hamilton-Jacobi formulation of reachable sets for continuous dynamic games," IEEE Trans. Autom. Control., **50**, No. 7, 947–957 (2005).
14. A. A. Agrachev and Yu. L. Sachkov, Geometric Control Theory [in Russian], Fizmatlit, Moscow (2005).
15. A. G. Chentsov, The Method of Programmed Iterations for an Approach-Evasion Differential Game [in Russian], Sverdlovsk (1973), Deposited in VINITI, No. 1933-79Dep.
16. N. N. Krasovskii, "An approach-evasion differential game. I," Izv. Akad. Nauk SSSR, Tekh. Kibern., No. 2, 3–18 (1973).